Resource letter TMD: Mathematics of Dark Energy

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Abstract and introduction

In physical cosmology and astronomy, dark energy is an unknown form of energy which is hypothesized to permeate all of space, tending to accelerate the expansion of the universe. Dark energy is the most accepted hypothesis to explain the observations since the 1990s indicating that the universe is expanding at an accelerating rate.

Assume that the standard model of cosmology is correct, the best current measurements indicate that dark energy contributes 68.3% of the total energy in the present-day observable universe. The mass–energy of dark matter and ordinary (baryonic) matter contribute 26.8% and 4.9%, respectively, and other components such as neutrinos and photons contribute a very small amount. The density of dark energy (7 × 10e-30 g/cm3) is very low, much less than the density of ordinary matter or dark matter within galaxies. However, it comes to dominate the mass–energy of the universe because it is uniform across space. (Wikipedia, 2011).

The story starts from Einstein who was trying to understand nature of the cosmos. And he was being told by the astronomers at the time that the universe was fixed and unchanging. Therefore, the distance between galaxies also remained constant. Accordingly, there might be local movements, such as the Sun goes around the Milk Way or the Earth goes around the Sun. But in the cosmic scale, the universe just remained unchanging.

This theory causes an immediate problem for Albert Einstein because he knew that the only long-scared force that was operating, was the gravitational force. All of the present galaxies would exert a gravitational force which was an attractive force on all the other galaxies. Therefore, those galaxies should be attracting themselves and thus pulling all the galaxies in. The universe would have been contracting if this fact had been true. However, the astronomers at the time were quite adamant that the cosmos was unchanging. Due to that, Einstein concluded that there had to be a force which he called by the cosmological constant. This force must be equal and opposite to the gravitational force in the universe. The combination of them essentially maintained the distances between galaxies in space. Before Einstein had understood very far with this problem, an astronomer named Edwin Hubble showed that the cosmos was in fact expanding in an exponential rate. Let's mathematically analyse this problem.

Mathematical analysis

Let D be the distance between two random galaxies in the universe. Let Δx be the difference in coordinate system in 1 dimensional space. Let a(t) be a function of time. It can be noted that

$$D = \Delta x \times a(t) \qquad (1)$$

Let's differentiate both sides of (1) – (respect to time)

$$\frac{dD}{dt} = \frac{d(\Delta x \times a(t))}{dt} \qquad (2)$$

The Δx remains unchanged, so it is a constant. But function a(t) of course does change with time.

$$\frac{dD}{dt} = \Delta x \frac{d(a(t))}{dt} \qquad (3)$$

Or equivalently,

$$V = \Delta x \frac{d(a(t))}{dt} \qquad (4)$$

Add a/a to the RHS of (4)

$$V = \Delta x \frac{d(a(t))}{dt} \times \frac{a}{a} \qquad (5)$$

According to (1), it can be noticed that

$$V = D \frac{d(a(t))}{dt} \times \frac{1}{a} = D \frac{a'}{a} = DK \qquad (6)$$

Where K is Hubble Constant. If you know the velocity of a receding galaxy, you can calculate how far away it is. It also implies that the further something is away from us, the faster it is travelling. Furthermore, Hubble Constant is a space constant, but not a time constant as it changes with time. We have a new look about the cosmos where all galaxies are moving away from one another. And there is no sense to conclude in which there is a centre of the universe. Let's rewrite the formulas to calculate the kinetic and potential energy of the moving galaxies.

$$KE = \frac{1}{2} m V^2 \quad and \quad PE = -\frac{GMm}{D} \qquad (7)$$

Where m and V is the mass and velocity of a galaxy A, M is the mass of a group of galaxies that contains A. G is simply the universal gravitational constant, D is the distance between A and the centre of the galaxies' group. The total mechanical energy is the sum of kinetic and potential energy. And its values must be a constant in time

$$ME = KE + PE = \frac{1}{2} m V^2 - \frac{GMm}{D} = k = constant \qquad (8)$$

Let's multiply both sides of (8) by 2, and divide both sides of (8) by m

$$\frac{1}{2} m V^2 - \frac{GMm}{D} = k$$

$$m V^2 - \frac{2GMm}{D} = 2k$$

$$V^2 - \frac{2GM}{D} = \frac{2k}{m} \qquad (9)$$

Where 2k/m is another constant because k and m are all constants. Let's k denote 2k/m.

$$V^2 - \frac{2GM}{D} = k \quad (10)$$

Rewrite (1) and (2)

$$D = \qquad (1)$$

$$\frac{dD}{dt} = V = \frac{d(\Delta x \times a(t))}{dt} = \Delta x \times a'(t) \qquad (2)$$

Substitute (1) and (2) into (10)

$$(\Delta x \times a'(t))^2 - \frac{2GM}{\Delta x \times a(t)} = k \qquad (11)$$

We know that mass of a group of galaxies is the multiplicative product of its density p and volume V. As the universe expands, volume V does increase, but mass does not. Then density therefore decreases. To be more precise, we have

$$M = pV = \frac{4}{3}\pi D^3 \times p(t) \qquad (12)$$

The density p will be performed as a function of time, as it varies with time. Substituting (12) into (11) gives us the following

$$(\Delta x \times a'(t))^2 - \frac{2G\left[\frac{4}{3}\pi D^3 \times p(t)\right]}{\Delta x \times a(t)} = k \qquad (13)$$

Then substitute (1) into (13)

$$(\Delta x \times a'(t))^2 - \frac{2G\left[\frac{4}{3}\pi(\Delta x \times a(t))^3 \times p(t)\right]}{\Delta x \times a(t)} = k \qquad (14)$$

Simplify (14)

$$(\Delta x \times a'(t))^2 - \frac{2G\left[\frac{4}{3}\pi(\Delta x \times a(t))^2 \times p(t)\right]}{1} = k \qquad (15)$$

In order to be dimensionally consistent, we can simplify Δx^2 because it is a constant. On the right hand side of (15), k will become a new constant.

$$(a'(t))^2 - \frac{2G\left[\frac{4}{3}\pi(a(t))^2 \times p(t)\right]}{1} = k \qquad (16)$$

Simplify (16) again

$$(a'(t))^2 - \left[\frac{8\pi G}{3}(a(t))^2 \times p(t)\right] = k \qquad (17)$$

Make $a'(t)^2$ the subject and then divide both sides of (17) by $a(t)^2$

$$\frac{(a'(t))^2}{(a(t))^2} = \left[\frac{8\pi G}{3} \times p(t)\right] - \frac{k}{(a(t))^2} \quad - convention \qquad (18)$$

And (18) is a simple version of Friedman Robertson Walker formula. LHS of (18) can be traced to the kinetic energy term, RHS of (18) can be traced back to the potential energy and a constant term. There is no denying that

$$\left[\frac{8\pi G}{3} \times p(t)\right] > 0 \qquad (19)$$

And if "-k" is also positive, then RHS and LHS of (18) will be positive. It indicates that universe constantly expand forever. That's called an open universe. If "-k" was negative and made LHS of (18) become negative, then the cosmos would eventually stop growing and start contracting. That would be called a closed universe. The last case is that k=0, then the kinetic energy is balanced to the potential energy term. That's called a flat universe. On simpler terms, as k=0, the universe will continue to expand but it will slow down.